4.5 Post-Newtonian order gravitational radiation

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*Marchand T., Blanchet L., Faye G. Class.Quant.Grav. 33 (2016) no.24, 244003 (**arxiv:1607.07601**)*

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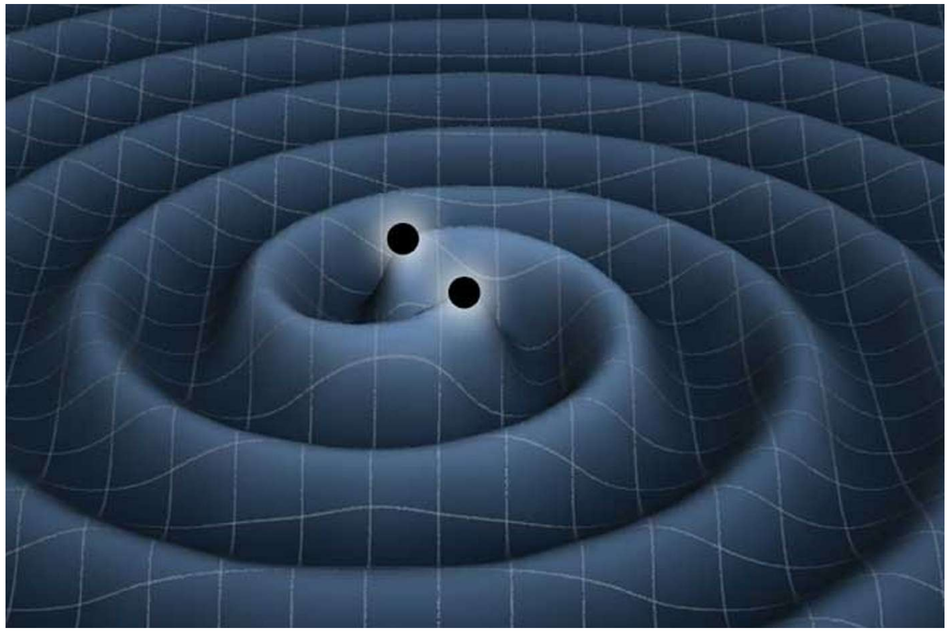
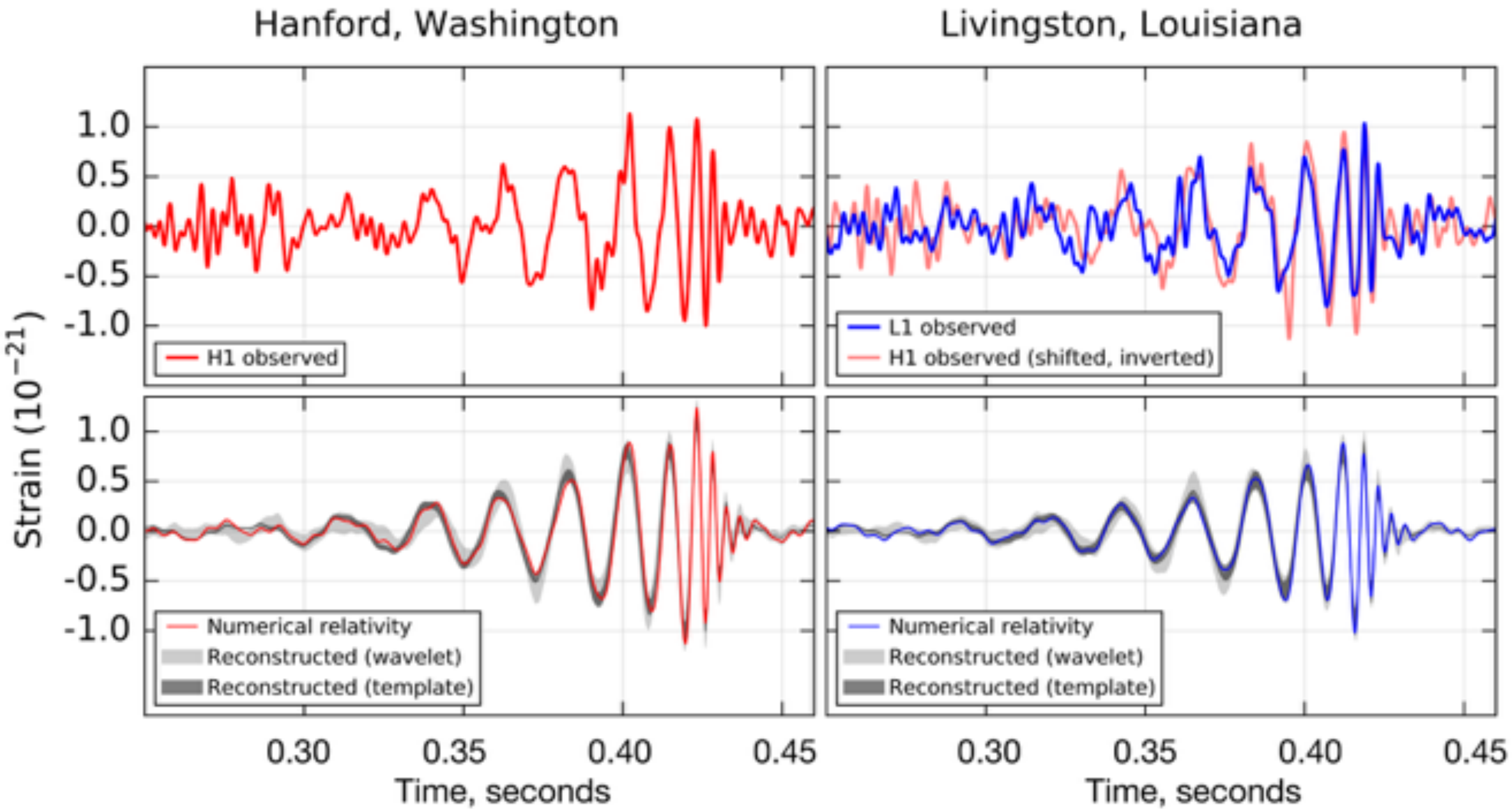
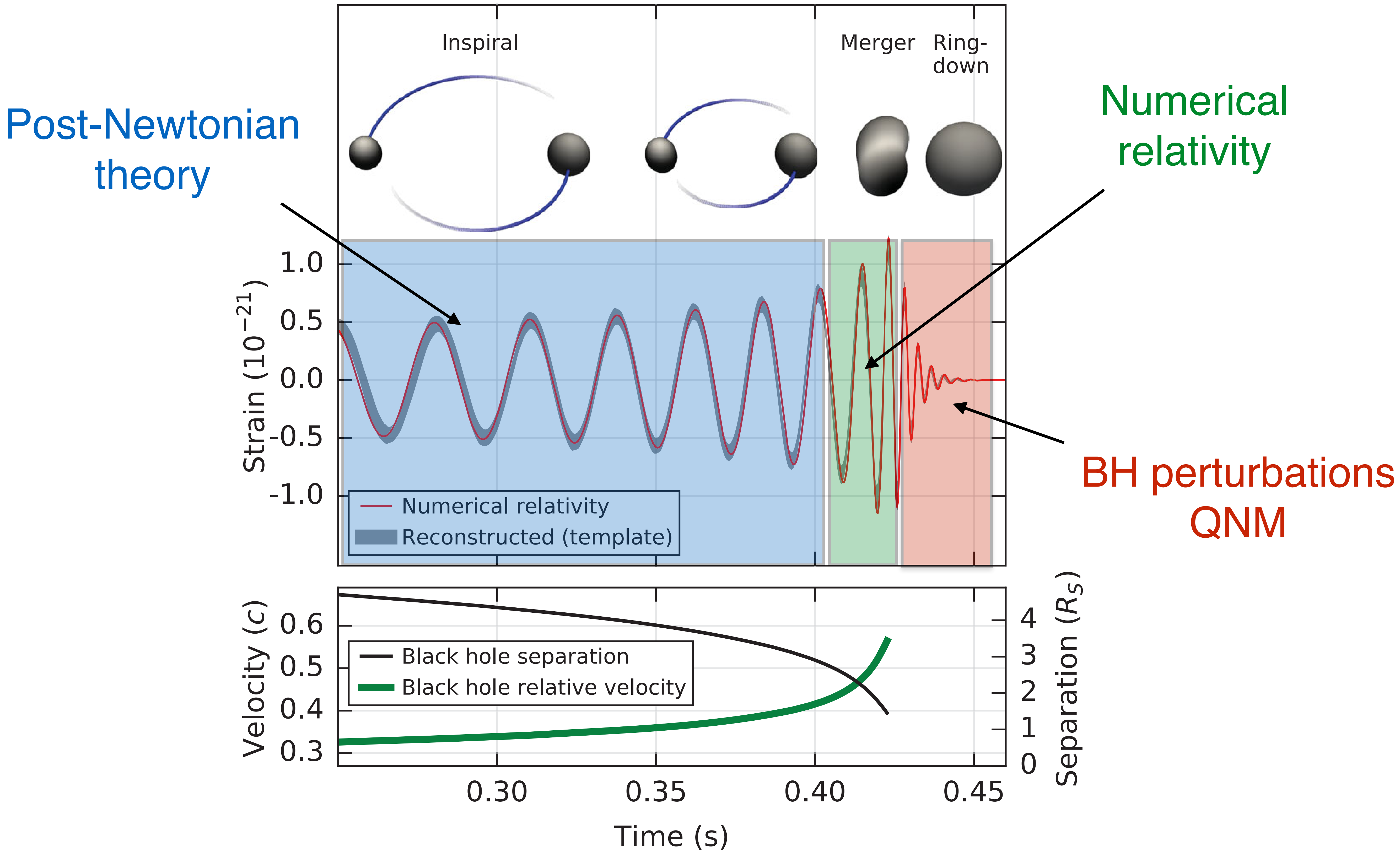
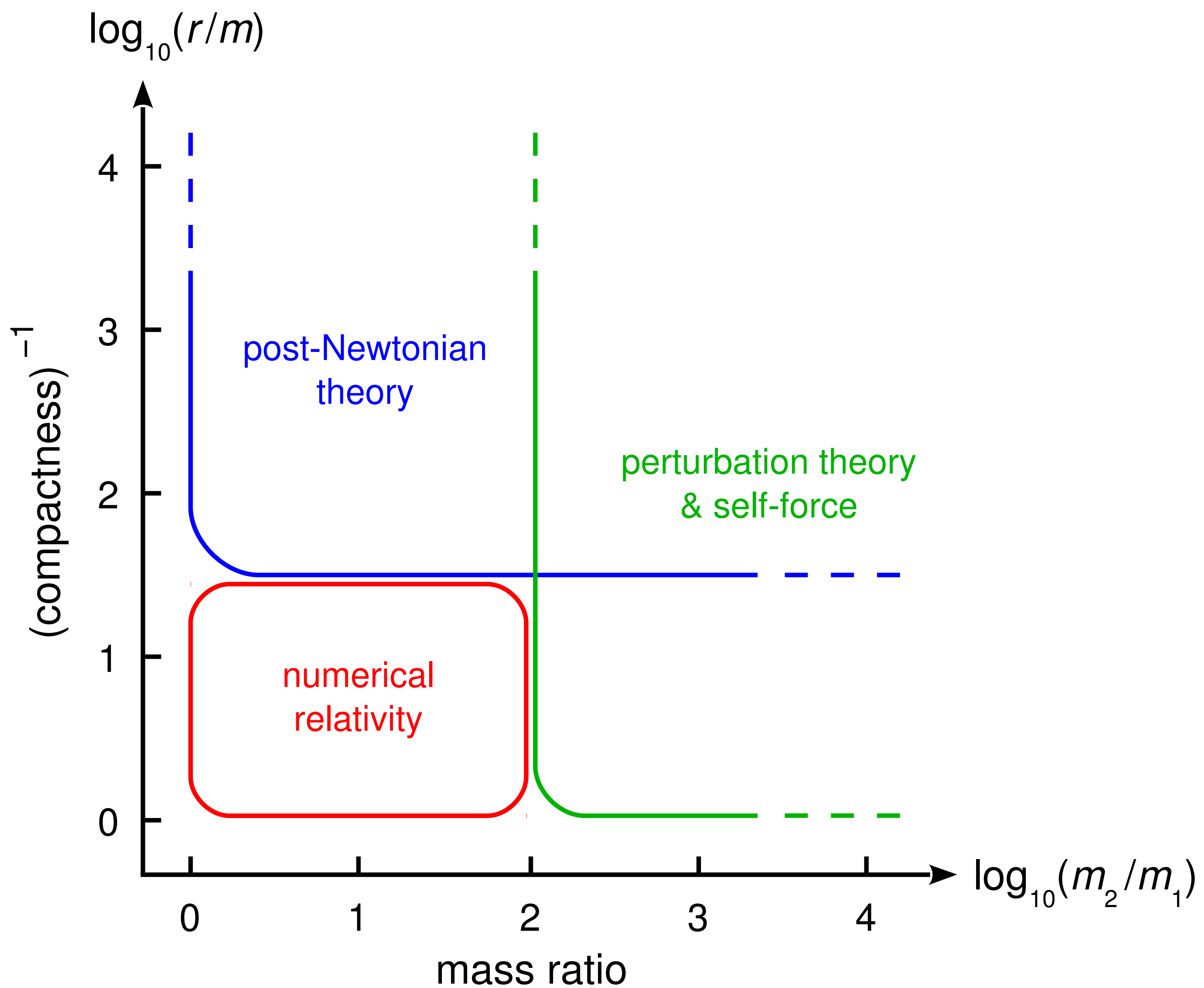
Introduction



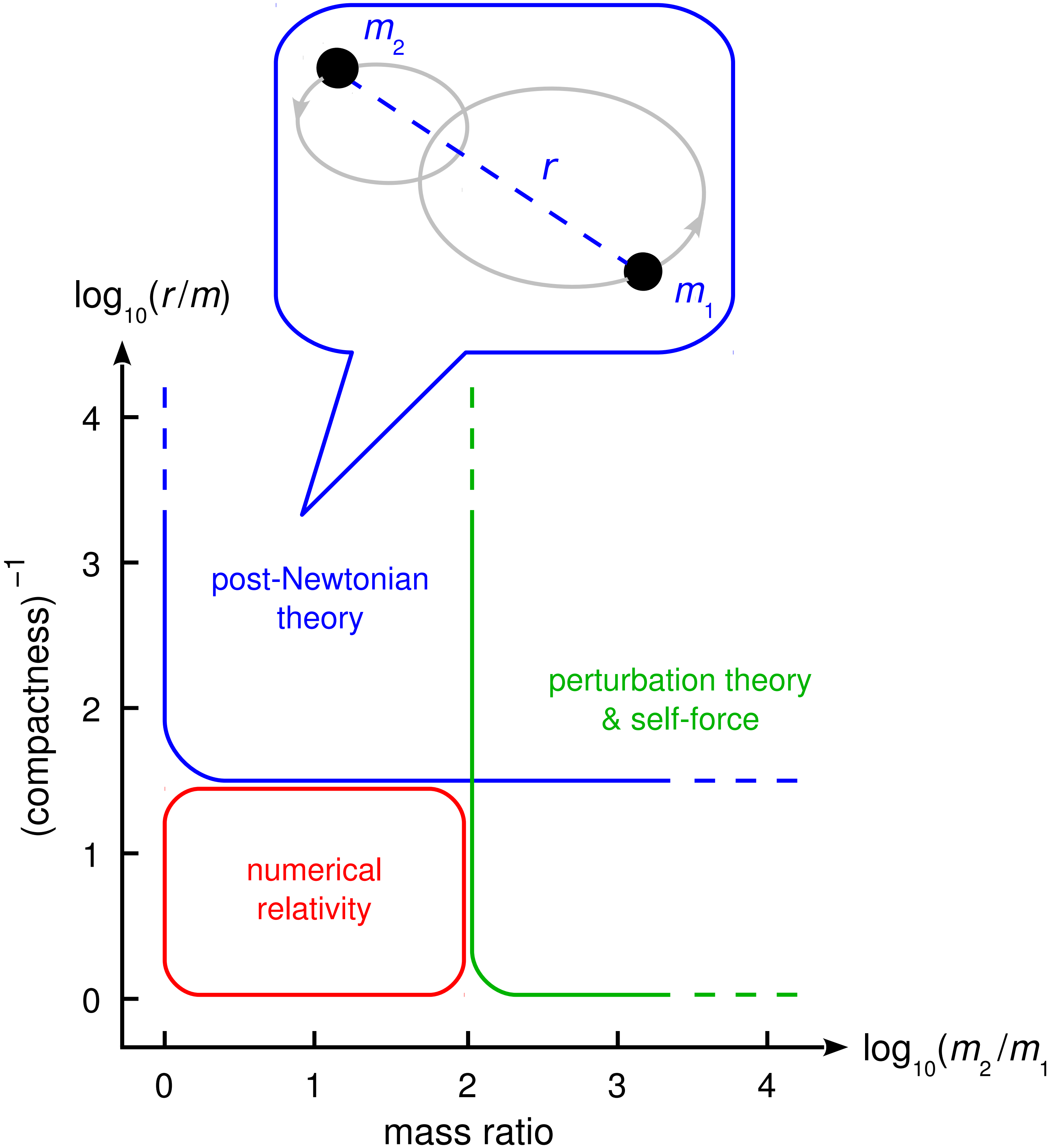




PRL 116, 061102 (2016)



[Courtesy of A. Le Tiec]



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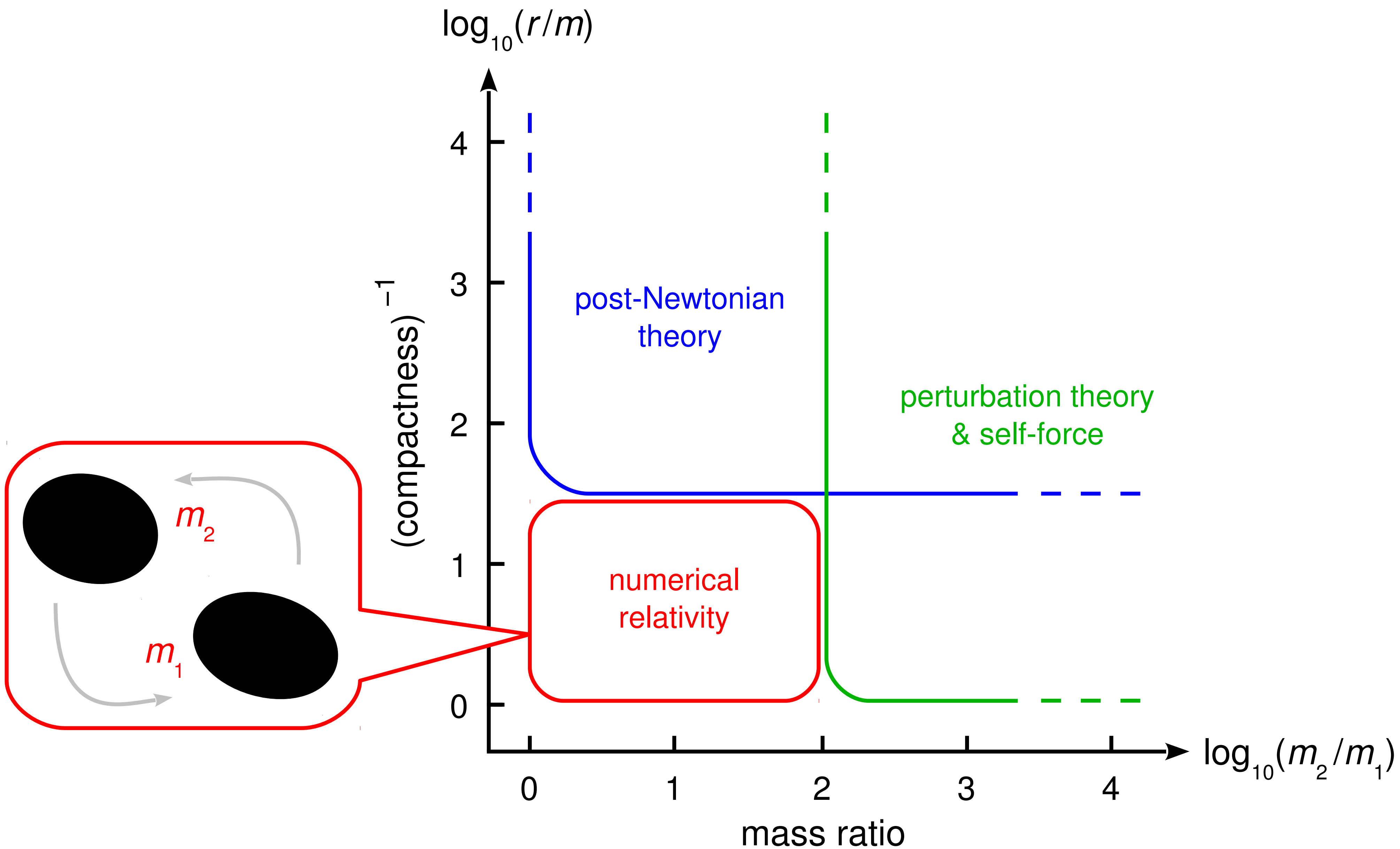

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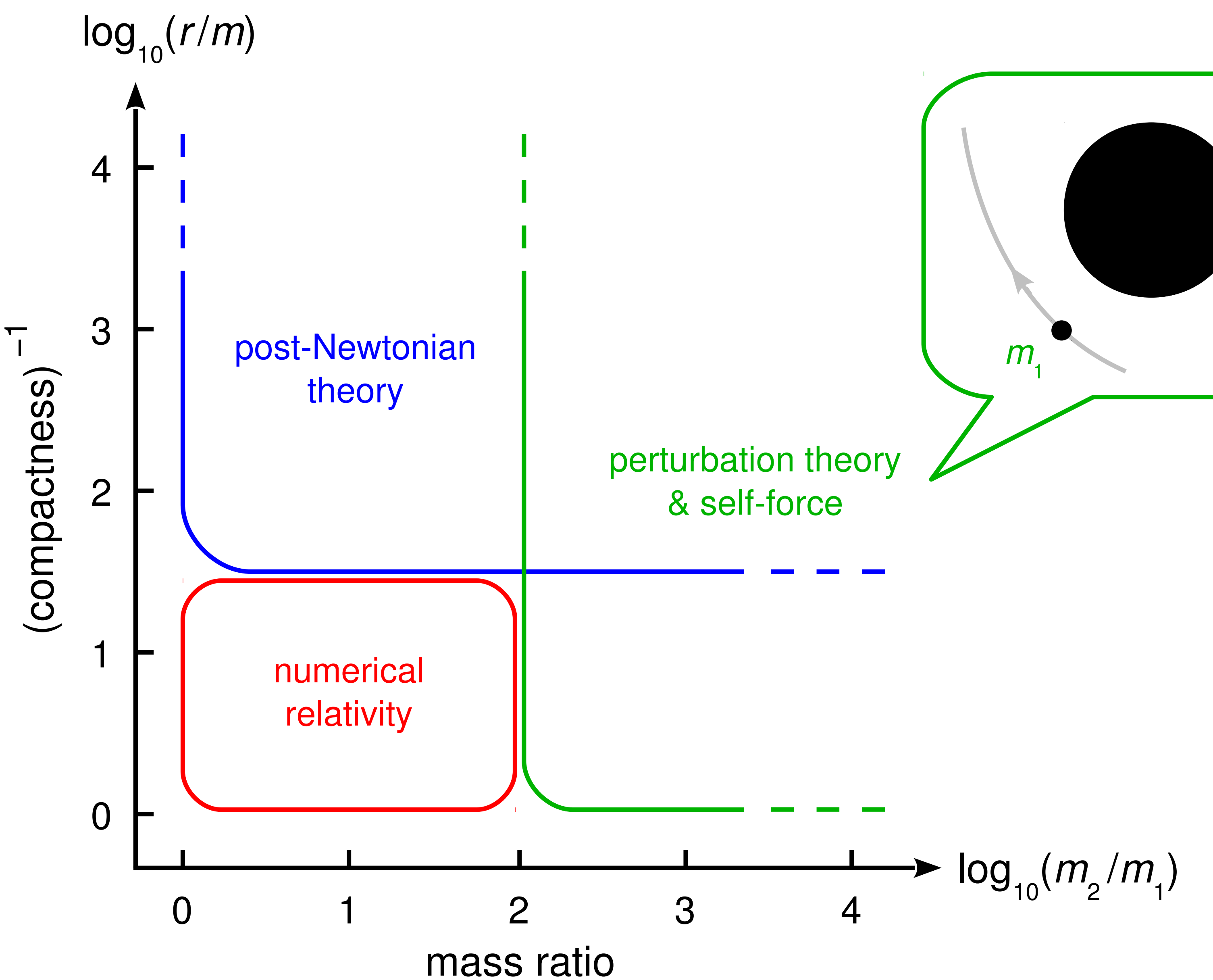

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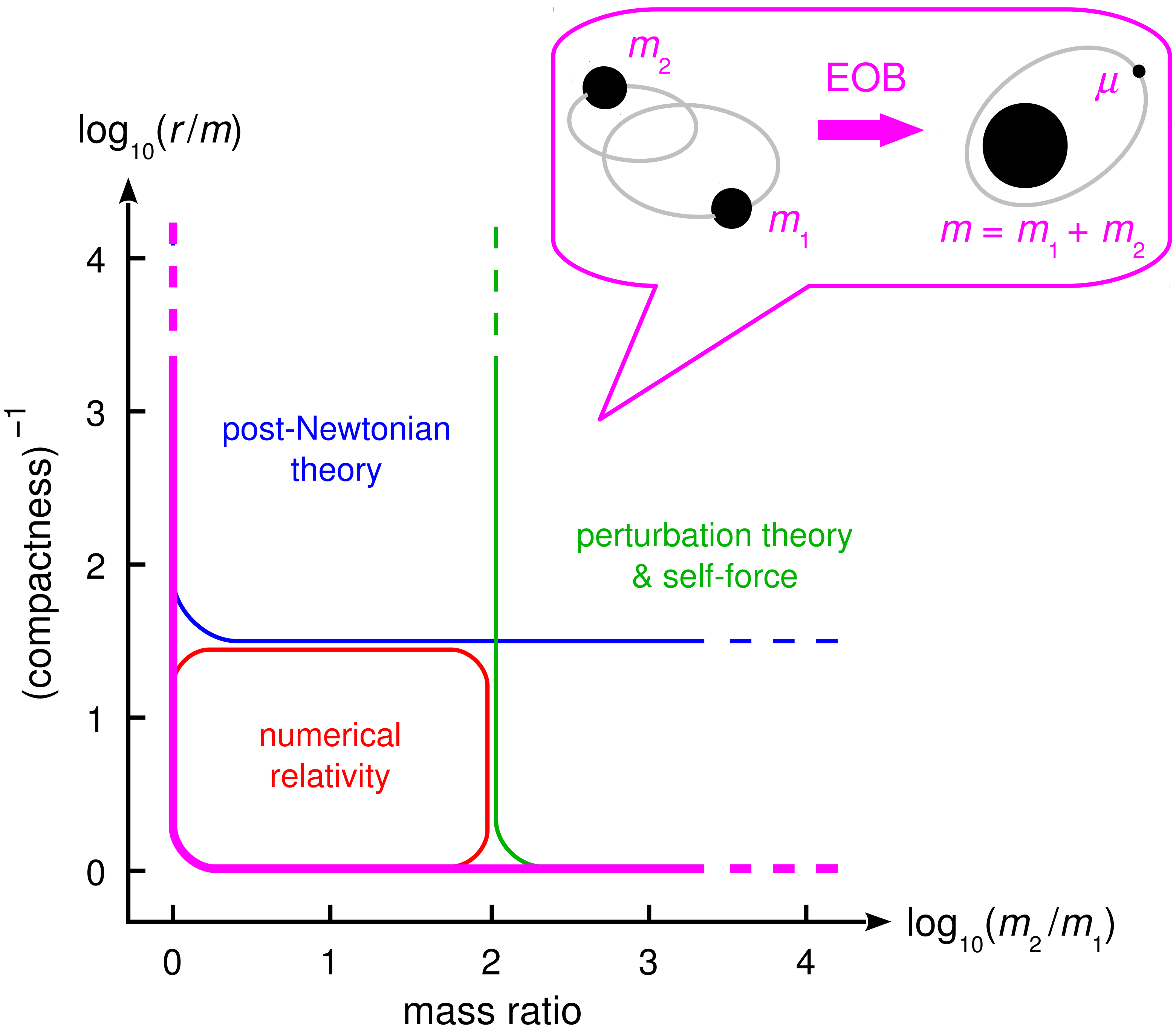
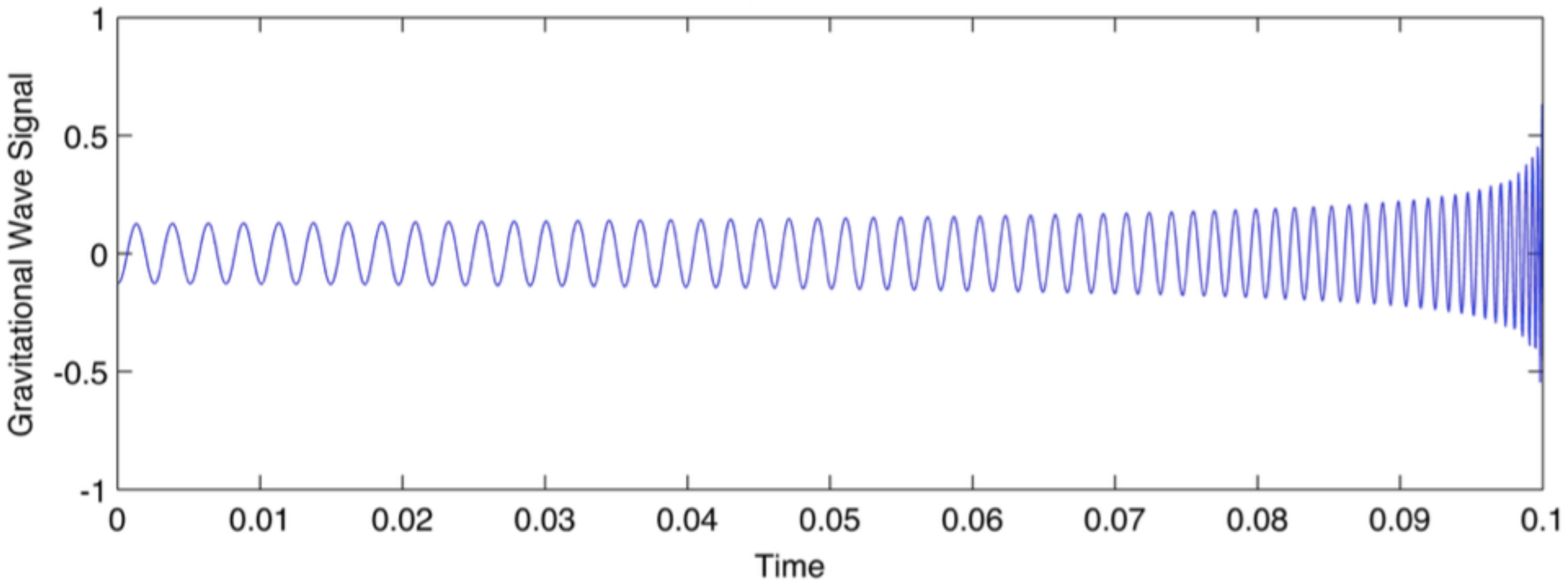


[Courtesy of A. Le Tiec]

How many cycles we detect?



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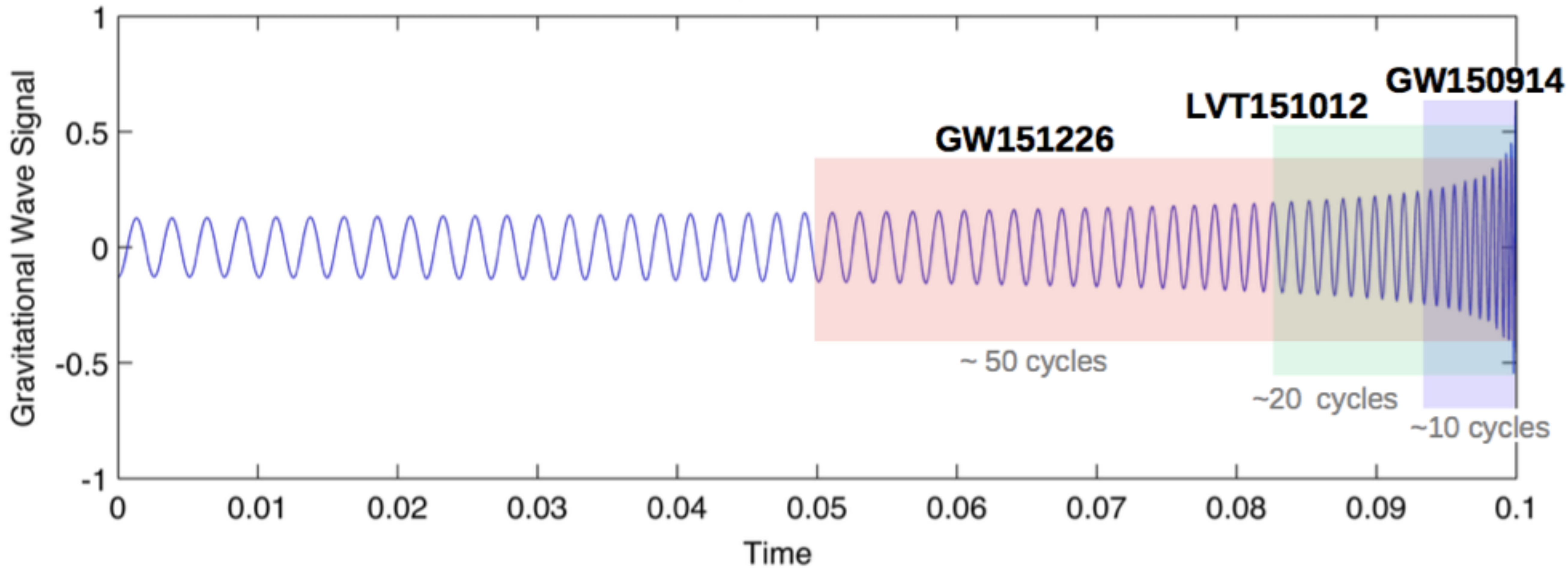
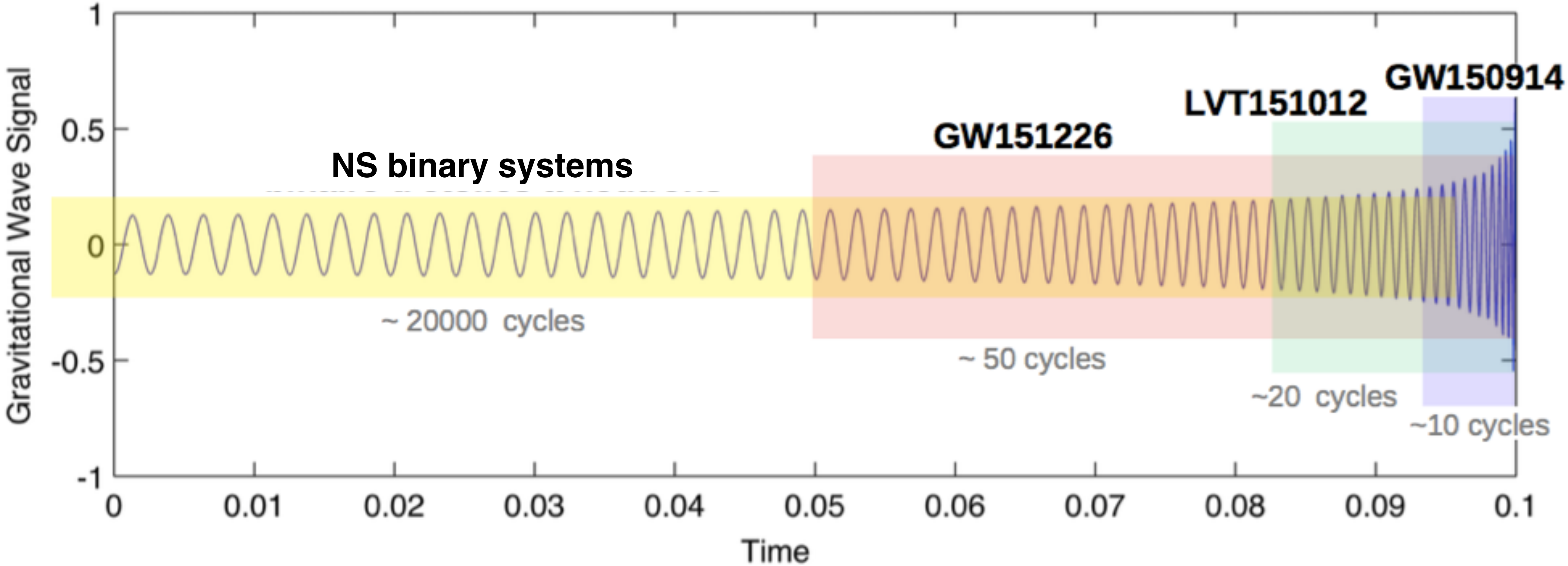
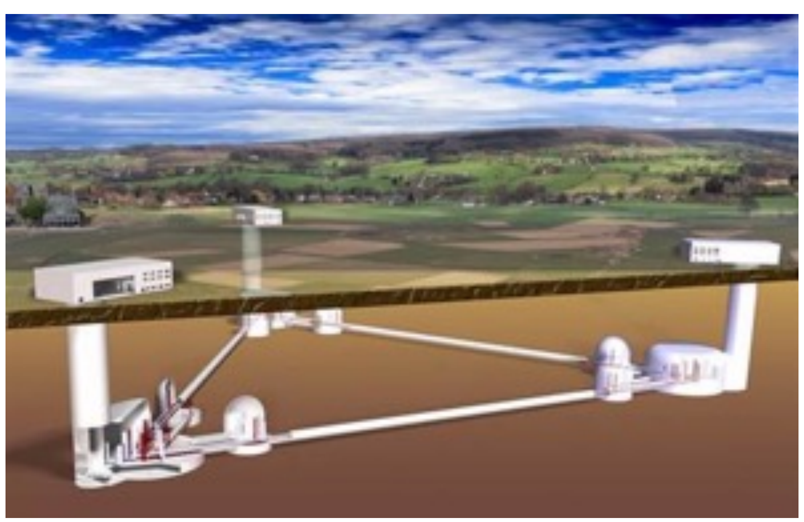


How many cycles we detect?



Einstein telescope



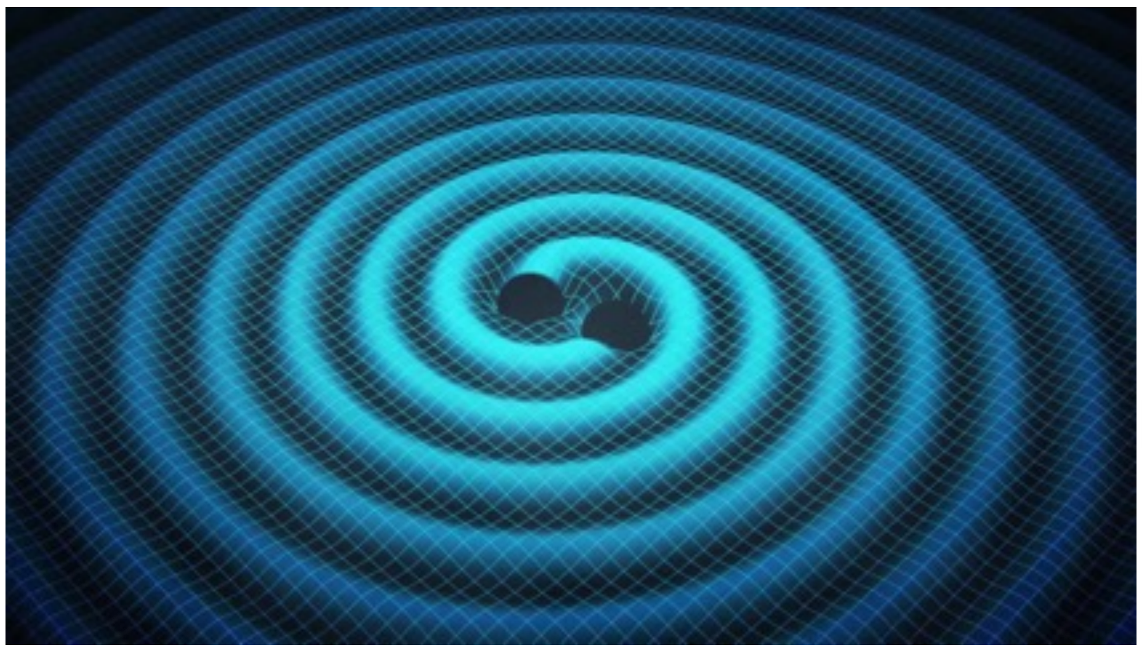
LISA

- Post-Newtonian theory
- The Multipolar-Post-Minkowskian algorithm
- Going to 4.5 PN
- Results

Post-Newtonian theory

Post-Newtonian theory

- Perturbative expansion of relativistic effects
- 1 PN $\longrightarrow \left(\frac{v}{c}\right)^2$



2 tasks:

Equation of motion

Emission and propagation of GW

Einstein quadrupole formula:

$$\mathcal{F} = \frac{\mathcal{G}}{5c^5}\left(\frac{\mathrm{d}^3 M_{ij}^{(N)}}{\mathrm{d}t}\frac{\mathrm{d}^3 M_{ij}^{(N)}}{\mathrm{d}t} + \mathcal{O}\left(\frac{1}{c^2}\right)\right)$$

$\Rightarrow$ **Blanchet-Damour-Iyer formalism**

Blanchet-Damour-Iyer formalism

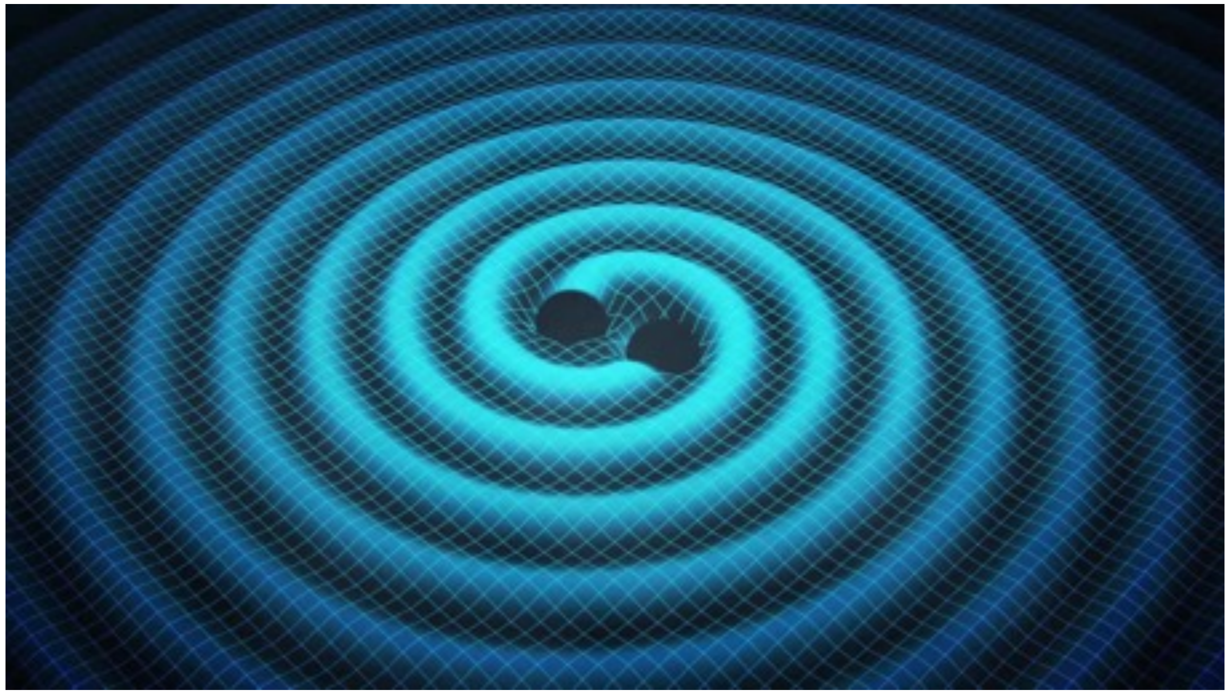
$$\frac{\text{Length of the source}}{\lambda_{GW}} \sim \frac{v}{c}$$

The source is post-Newtonian if: $\frac{v}{c} << 1$

If the source is post-Newtonian we can define:

- A near zone: $r << \lambda$
- A far zone: $r >> \text{Length of the source}$
- An intermediate zone: $\text{Length of the source} << r << \lambda$

Blanchet-Damour-Iyer formalism



- PN expansion in the near zone
- Multipolar post-Minkowkian expansion in the far zone
- Matching equation between the two expansions

- Post-Newtonian development diverges in the far-zone

$$F(t-\frac{r}{c}) = F(t) - \frac{r}{c}F'(t) + \ldots \quad \text{diverges for} \quad r \to \infty$$

- (Vacuum) multipolar expansion diverges when $r \to 0$, and is not valid inside the source
- If the source is post-Newtonian, there is an intermediate zone for which both expansions are valid and where a matching equation can be written [[…];Blanchet 98]

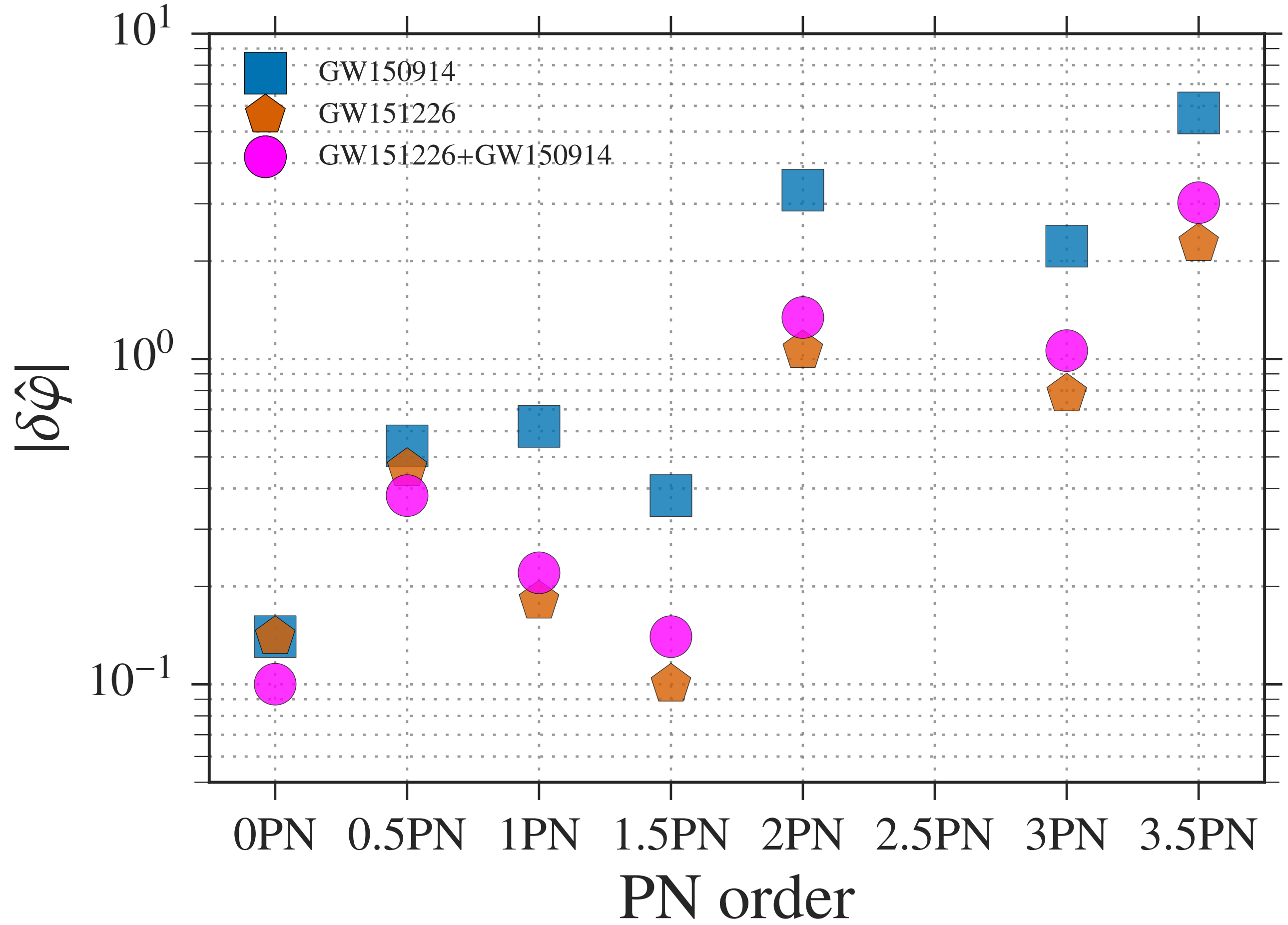
Once the **equations of motions** and the **flux** is known at n-PN, we consider circular orbits:

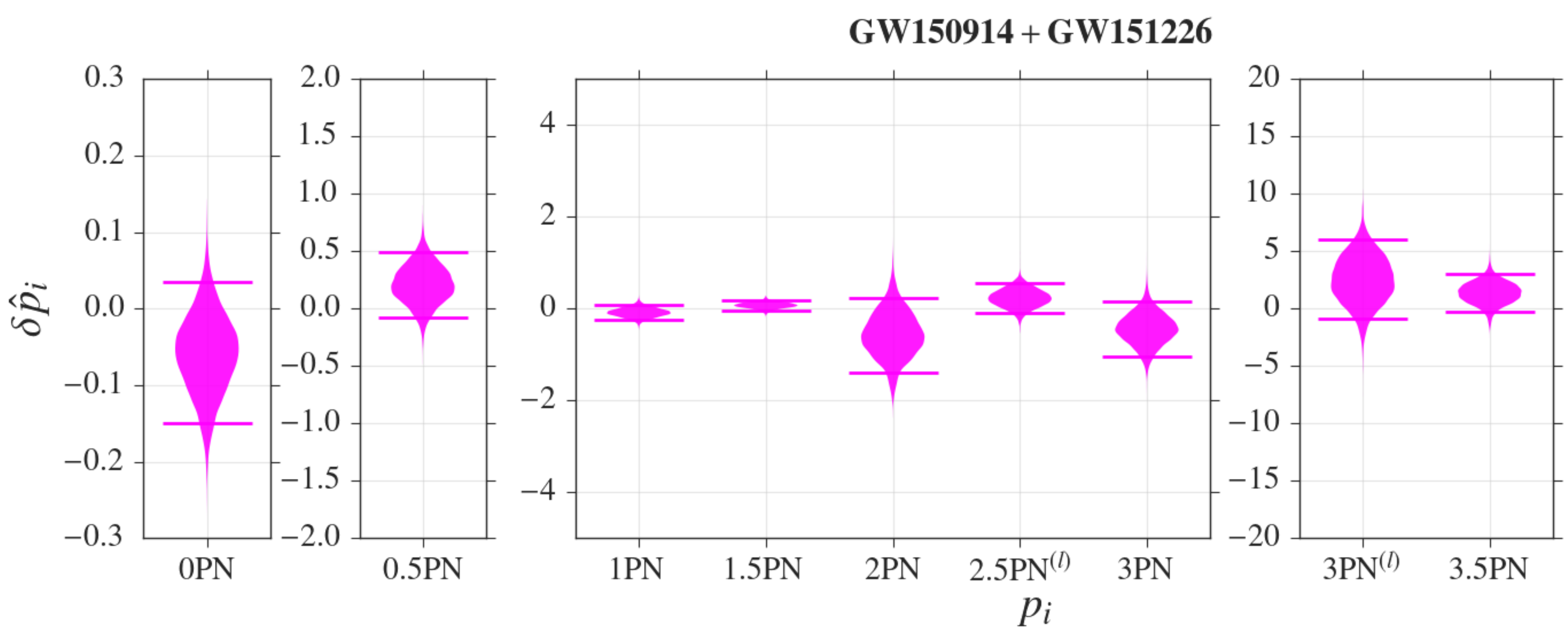
$$\frac{\mathrm{d}E_\mathrm{n}}{\mathrm{d}t} = \mathcal{F}_\mathrm{n} \Rightarrow \phi = \int \Omega(t)\mathrm{d}t = \int \frac{\mathrm{d}\Omega}{\dot{\Omega}}\Omega = \int \mathrm{d}\Omega \frac{\frac{\mathrm{d}E}{\mathrm{d}\Omega}}{F(\Omega)}\Omega$$

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$x = \left(\frac{Gm\Omega}{c^3}\right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2}\right)$$

$$\begin{aligned}
\phi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right) \\
& + \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{856}{21}\ln(16\,x)\right. \\
& \quad \left. + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\right]x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},
\end{aligned}$$





LIGO Scientific and Virgo collaboration arxiv:1606.04856

Impact of 4.5PN

Post-Newtonian contributions to the accumulated number of gravitational-wave cycles N_{cycle} for compact binaries detectable in the bandwidth of LIGO-VIRGO detectors. The entry frequency is $f_{seismic}$ = 10 Hz and the terminal frequency is the ISCO frequency.

PN order		$1.4\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 10\,M_\odot$
N	(inst)	15952.6	3558.9	598.8
1PN	(inst)	439.5	212.4	59.1
1.5PN	(leading tail)	−210.3	−180.9	−51.2
2PN	(inst)	9.9	9.8	4.0
2.5PN	(1PN tail)	−11.7	−20.0	−7.1
3PN	(inst + tail-of-tail)	2.6	2.3	2.2
3.5PN	(2PN tail)	−0.9	−1.8	−0.8

[Liv. Review L. Blanchet]

The Multipolar-Post-Minkowskian algorithm

The Multipolar-Post-Minkowskian algorithm

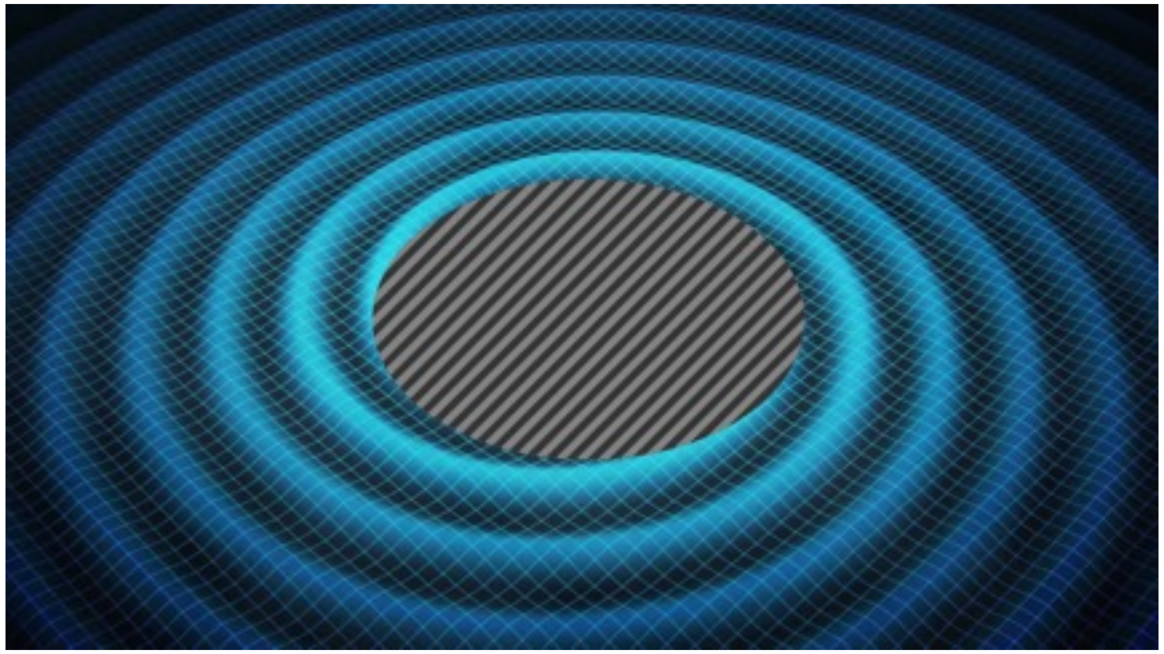
- **Goal**: building the most general past-stationary vacuum solution representing the spacetime in the far-zone

- This solution will depend on unspecified functions (M_L and S_L).

- The matching equation will relate these functions to the physical properties of the source and will provide explicit formulae to compute them.

- M_L correspond to the mass L-multipole moments of the source and S_L to its current L-multipole moments.

The MPM algorithm

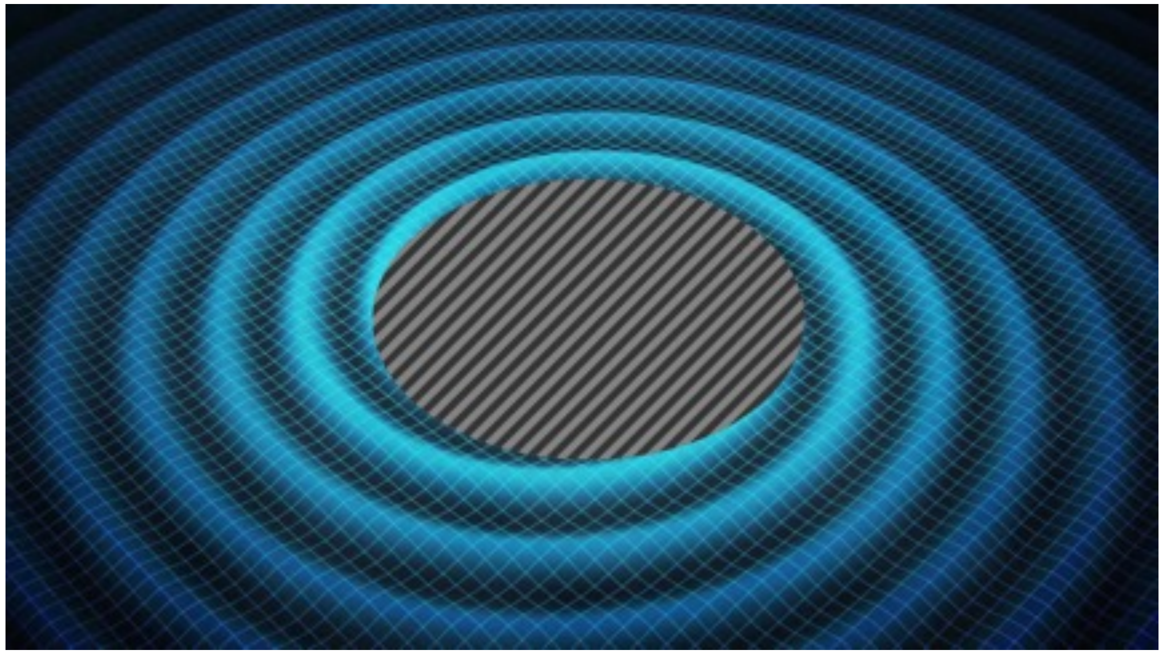


$$G_{\mu\nu}(g_{\alpha\beta}, \partial g_{\alpha\beta}, \partial^2 g_{\alpha\beta}) = 0$$

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} = \mathcal{G}h^{\mu\nu} + \mathcal{G}^2 h^{(2)\mu\nu} + \ldots$$

$$\begin{cases} \Box h^{(i)}_{\mu\nu} = \Lambda_{\mu\nu}(h^{(1)}, \ldots, h^{(i-1)}) \\ \partial^\mu h^{(i)}_{\mu\nu} = 0 \end{cases}$$

The MPM algorithm



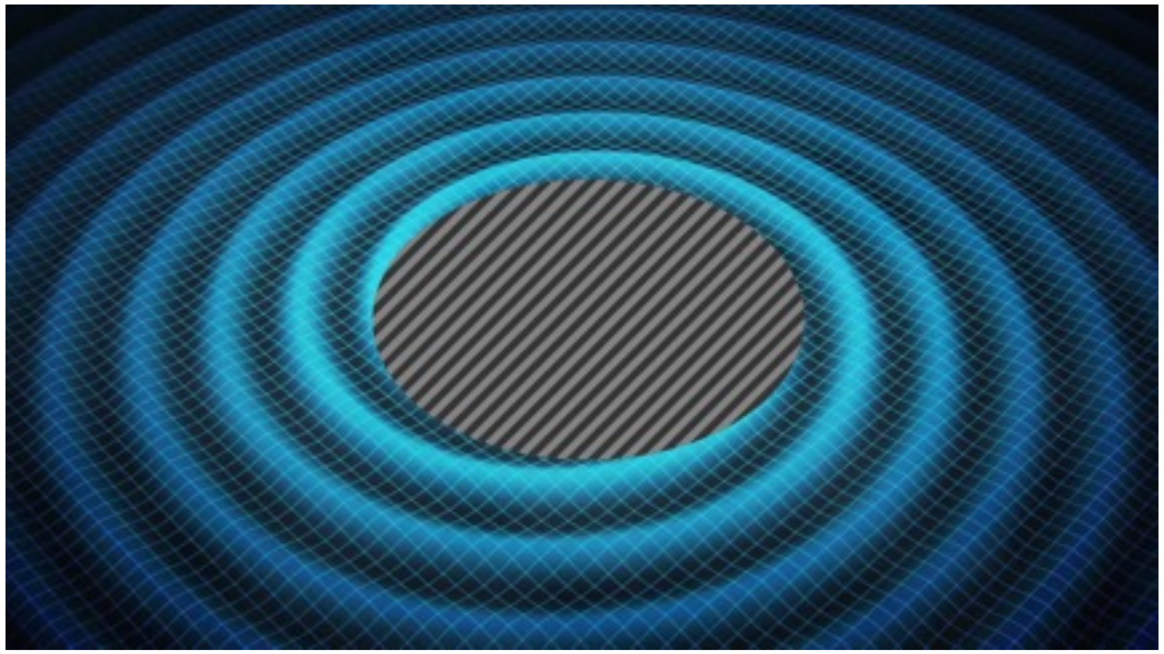
$$\begin{cases} \Box h^{(i)}_{\mu\nu} = \Lambda_{\mu\nu}(h^{(1)}, \ldots, h^{(i-1)}) \\ \partial^\mu h^{(i)}_{\mu\nu} = 0 \end{cases}$$

$$h^1_{\mu\nu} \sim \sum_{l\geq 0} \partial_{i_1,\ldots,i_l} \left(\frac{M_{i_1\ldots i_l}(t-r)}{r} \right) + \sum \partial_{i_1,\ldots,i_l} \left(\frac{S_{i_1\ldots i_l}(t-r)}{r} \right)$$
$$= h^1_M + h^1_{M_{ij}} + h^1_{M_{ijk}} + \cdots + h^1_{S_{ij}} + h^1_{S_{ijk}} + \ldots$$

[Thorne 80]

$$h^{(2)} = h^{(2)}_{M\times M} + h^{(2)}_{M\times M_{ij}} + h^{(2)}_{M_{ij}\times M_{ij}} + \ldots$$

First issue: UV regularisation



$$\begin{cases} \Box h^{(i)}_{\mu\nu} = \Lambda_{\mu\nu}(h^{(1)}, \ldots, h^{(i-1)}) \\ \partial^\mu h^{(i)}_{\mu\nu} = 0 \end{cases}$$

$$\Box^{-1}\Lambda(x,t) = \int \mathrm{d}^3x' \frac{\Lambda(x', t - \mid x - x' \mid)}{\mid x - x' \mid}$$

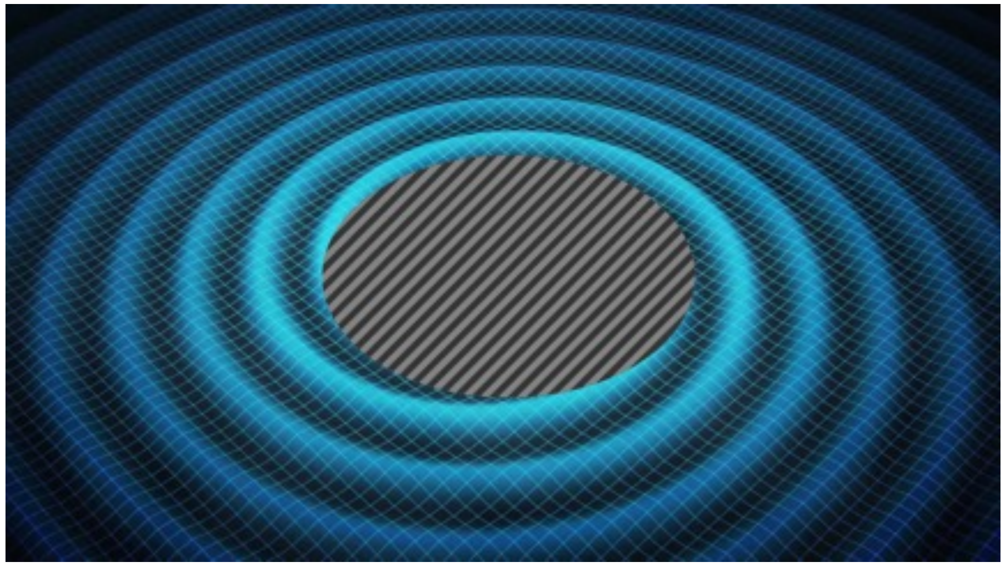
Issue: $\Lambda \sim_{r\to 0} \frac{1}{r^k}, \; k \geq 3$

$$\mathrm{FP}_{B=0}\Box^{-1}\left[\left(\frac{r}{r_0}\right)^B \Lambda\right]$$

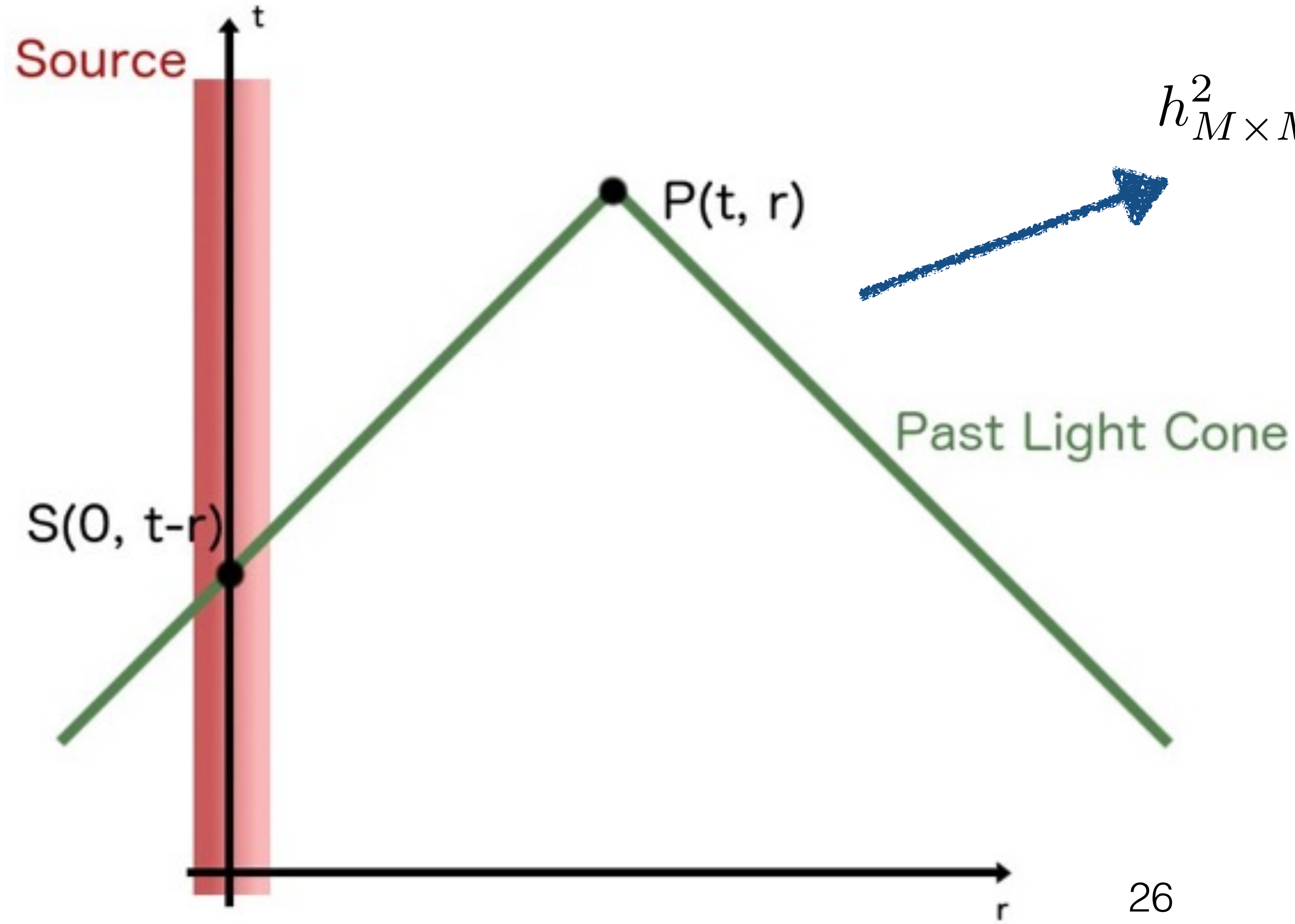
$B \in \mathbb{C}$, analytic continuation

$$\mathrm{FP}_{B=0}\left[\sum_{k\geq -k_0} g_k B^k\right] \equiv g_0$$

Second issue: tails



$$\begin{cases} \Box h^{(i)}_{\mu\nu} = \Lambda_{\mu\nu}(h^{(1)}, \ldots, h^{(i-1)}) \\ \partial^\mu h^{(i)}_{\mu\nu} = 0 \end{cases}$$



$$h^2_{M\times M_{ij}}(t,r) \sim M \int_0^\infty \mathrm{d}\tau M_{ij}(t-r-\tau)\mathcal{Q}(\tau)$$

Tails

$$u(x,t) = \Box^{-1}_{\text{ret}}[S(x',t')] = -\frac{1}{4\pi}\int \frac{\mathrm{d}^3x'}{\mid x - x' \mid} S\left(x', t - \frac{\mid x' - x \mid}{c}\right)$$

If $S_L(x,t) = \hat{n}_L \tilde{S}\left(r, t - \frac{r}{c}\right)$ with $\hat{n}_L = \mathrm{STF}[n_{i_1} \cdots n_{i_l}]$

$$R(r,s) = r^l \int_0^r \mathrm{d}x \frac{(r-x)^l}{l!} \left(\frac{2}{x}\right)^{l-1} \tilde{S}(x,s)$$

$$u_L(x,t) = \int_{-\infty}^{t-r} \mathrm{d}s \hat{\partial}_L \left[\frac{R\left(\frac{t-r-s}{2}, s\right) - R\left(\frac{t+r-s}{2}, s\right)}{r}\right]$$

$$\Box^{-1}_{\mathrm{ret}}\left[\left(\frac{r}{r_0}\right)^B\frac{\hat{n}_L}{r^2}F(t-r)\right]=\frac{1}{2(2r_0)^B B(B-1)\cdots(B-l)}\int_r^\infty \mathrm{d}s F(t-s)\partial_L\frac{(s-r)^{B+l}-(s+r)^{B+l}}{r}$$

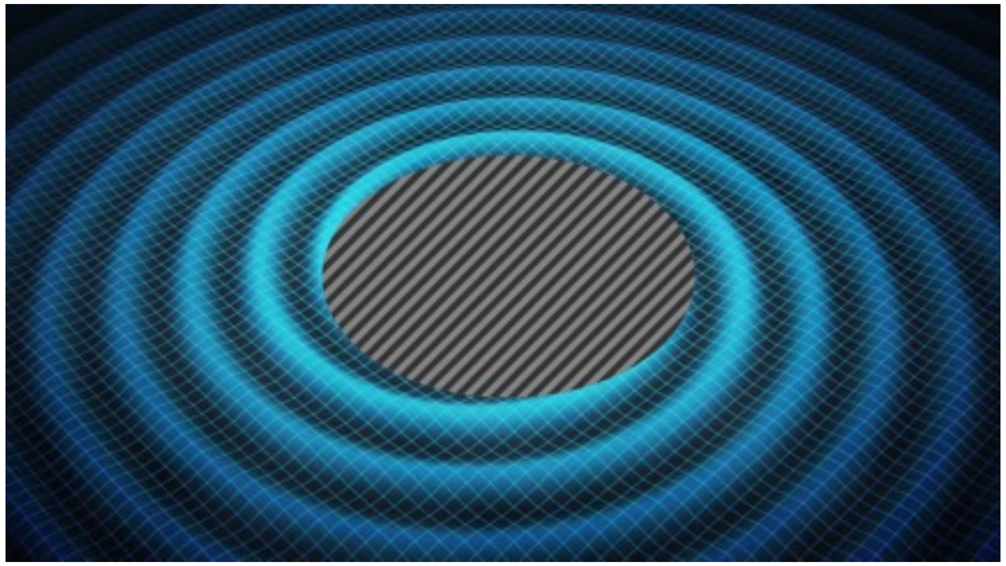
$$\mathrm{FP}_{B=0}\ \Box^{-1}_{\mathrm{ret}}\left[\left(\frac{r}{r_0}\right)^B\frac{\hat{n}_L}{r^2}F(t-r)\right]=\frac{(-1)^l}{2}\int_r^\infty \mathrm{d}s F(t-s)\partial_L\frac{(s-r)^{l}\ln(s-r)-(s+r)^{l}\ln(s+r)}{r}$$

$$\mathrm{FP}_{B=0}\ \Box^{-1}_{\mathrm{ret}}\left[\left(\frac{r}{r_0}\right)^B\frac{\hat{n}_L}{r^2}F(t-r)\right]=\frac{-\hat{n}_L}{r}\int_r^\infty \mathrm{d}s F(t-s)Q_l\left(\frac{s}{r}\right)$$

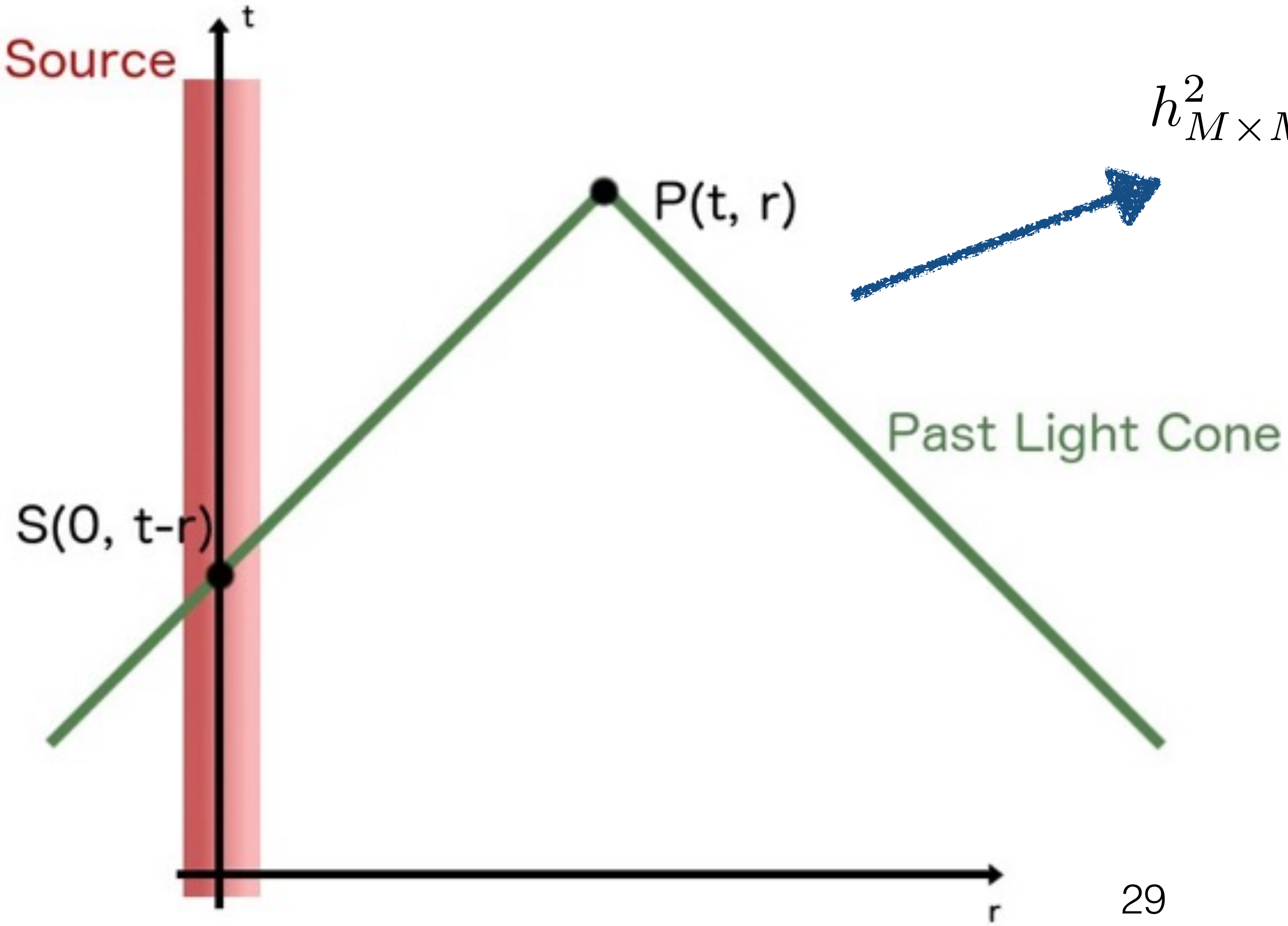
$$Q_l(x)=\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)-\sum_{j=1}^{l}P_{l-j}(x)P_{j-1}(x)$$

This enables to solve: $h^{(2)}_{\mu\nu}=\mathrm{FP}_{B=0}\Box^{-1}\Lambda_{\mu\nu}(h^{(1)\alpha\beta},h^{(1)\alpha\beta})$

Second issue: tails



$$\begin{cases} \Box h^{(i)}_{\mu\nu} = \Lambda_{\mu\nu}(h^{(1)}, \dots, h^{(i-1)}) \\ \partial^\mu h^{(i)}_{\mu\nu} = 0 \end{cases}$$



$$h^2_{M\times M_{ij}}(t,r) \sim M \int_0^\infty \mathrm{d}\tau M_{ij}(t-r-\tau)\mathcal{Q}(\tau)$$

The 4.5PN project

4.5PN project

Our goal: compute the phase at 4.5PN for circular orbit

- Equation of motion at 4PN
- Energy at 4.5PN
- Flux at 4.5PN: - Contributions to the tails at 4.5PN
  - Other contributions at 4PN which require to compute the 4PN mass quadrupole and the 3PN current quadrupole […]

In this presentation: contributions at 4.5PN of the tails:

$$h^{(2)}_{M\times M_{ij}}, h^{(3)}_{M^2\times M_{ij}} h^{(4)}_{M^3\times M_{ij}}$$

arxiv:1607.07601

▸ Required new analytical formulae

$$\mathrm{FP}_{B=0}\Box^{-1}\left[\hat{n}_L\left(\frac{r}{r_0}\right)^B r^{-k}\int_1^\infty \mathrm{d}y Q_m(y)F(t-ry)\right]$$
$$= -\hat{n}_L\int_1^\infty \mathrm{d}s F^{(k-2)}(t-rs)\left(Q_l(s)\int_1^s \mathrm{d}y Q_m^{(-k+2)}(y)P_l(y)+P_l(s)\int_s^\infty \mathrm{d}y Q_m^{(-k+2)}(y)Q_l(y)\right)$$

$$\mathrm{FP}_{B=0}\Box^{-1}\left[\hat{n}_L\left(\frac{r}{r_0}\right)^B \ln\left(\frac{r}{r_0}\right)\frac{F(t-r)}{r^2}\right]$$
$$= -\frac{\hat{n}_L}{2r}\int_r^\infty \mathrm{d}s F(t-s)Q_m\left(\frac{s}{r}\right)\left[\ln\left(\frac{s^2-r^2}{4r_0^2}\right)+2H_l\right]$$

▸ Implementing the algorithm into Mathematica

➡ $h^2_{M\times M_{ij}}$, $h^3_{M\times M\times M_{ij}}$, $h^4_{M\times M\times M\times M_{ij}}$

Going to radiative coordinate

When $r \to \infty$:

$$h^{00}_{M^3\times M_{ij}} = \frac{M^3 \hat{n}_{ab}}{r} \int_0^{+\infty} \mathrm{d}\tau\, M^{(6)}_{ab} \left\{ -\frac{8}{3}\ln^3\left(\frac{\tau}{2r}\right) + \frac{148}{21}\ln^2\left(\frac{\tau}{2r}\right) + \frac{232}{21}\ln\left(\frac{r}{r_0}\right)\ln\left(\frac{\tau}{2r}\right) + \frac{1016}{2205}\ln\left(\frac{\tau}{2r}\right) + \frac{104}{15}\ln\left(\frac{r}{r_0}\right) + \frac{16489}{1575} - \frac{232\pi^2}{63} \right\} + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right)$$

$$h^{0i}_{M^3\times M_{ij}} = \dots$$

$$h^{ij}_{M^3\times M_{ij}} = \dots$$

$$X^\mu = x^\mu + \xi^\mu(x) \qquad \begin{cases} \xi^0 = -2M\ln\left(\frac{r}{b_0}\right) \\ \xi^i = 0 \end{cases}$$

$$T = X^0$$

$$R = X^i X_i$$

Now $T - R = \mathrm{Const}$ represents true Light-cone

$$H^{00}_{M^3\times M_{ij}} = \frac{M^3 \hat{N}_{ab}}{R} \int_0^{+\infty} \mathrm{d}\tau\, M^{(6)}_{ab} \left\{ -\frac{8}{3}\ln^3\left(\frac{\tau}{2b_0}\right) - 4\ln^2\left(\frac{\tau}{2b_0}\right) + \frac{232}{21}\ln\left(\frac{\tau}{2b_0}\right)\ln\left(\frac{\tau}{2r_0}\right) - \frac{14272}{2205}\ln\left(\frac{\tau}{2b_0}\right) + \frac{104}{15}\ln\left(\frac{\tau}{2r_0}\right) + \frac{16489}{1575} - \frac{232\pi^2}{63} \right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Computing the flux for circular orbits

$$H^{00}_{M^3\times M_{ij}} = \frac{M^3\hat{N}_{ab}}{R}\int_0^{+\infty} \mathrm{d}\tau\, M^{(6)}_{ab}\left\{-\frac{8}{3}\ln^3\left(\frac{\tau}{2b_0}\right) - 4\ln^2\left(\frac{\tau}{2b_0}\right) + \frac{232}{21}\ln\left(\frac{\tau}{2b_0}\right)\ln\left(\frac{\tau}{2r_0}\right) - \frac{14272}{2205}\ln\left(\frac{\tau}{2b_0}\right) + \frac{104}{15}\ln\left(\frac{\tau}{2r_0}\right) + \frac{16489}{1575} - \frac{232\pi^2}{63}\right\} + \mathcal{O}\left(\frac{1}{R^2}\right), \qquad (4.4a)$$

$$H^{\mathrm{TT}}_{ij} = -\frac{4}{R}\mathcal{P}_{ijkl}\sum_{\ell=2}^{+\infty}\frac{1}{\ell!}\left\{N_{L-2}\,U_{klL-2}(T_R) - \frac{2\ell}{\ell+1}N_{aL-2}\,\varepsilon_{ab(k}\,V_{l)bL-2}(T_R)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

$$\mathcal{F} = \sum_{l=2}^{\infty}\frac{G}{c^{2l+1}}\left[a_l\left(U_L^{(1)}\right)^2 + \frac{b_l}{c^2}\left(V_L^{(1)}\right)^2\right]$$

- M_{ij} computed explicitly for circular orbits with the matching equation !

Computing the mass quadrupole

- $\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}\left[h, \partial h, \partial^2 h\right]$
- $\Sigma = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2}$, $\Sigma_i = \frac{\overline{\tau}_{0i}}{c}$, $\Sigma_{ij} = \overline{\tau}_{ij}$ (PN expansion)
- $\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!}(1-z^2)^l$

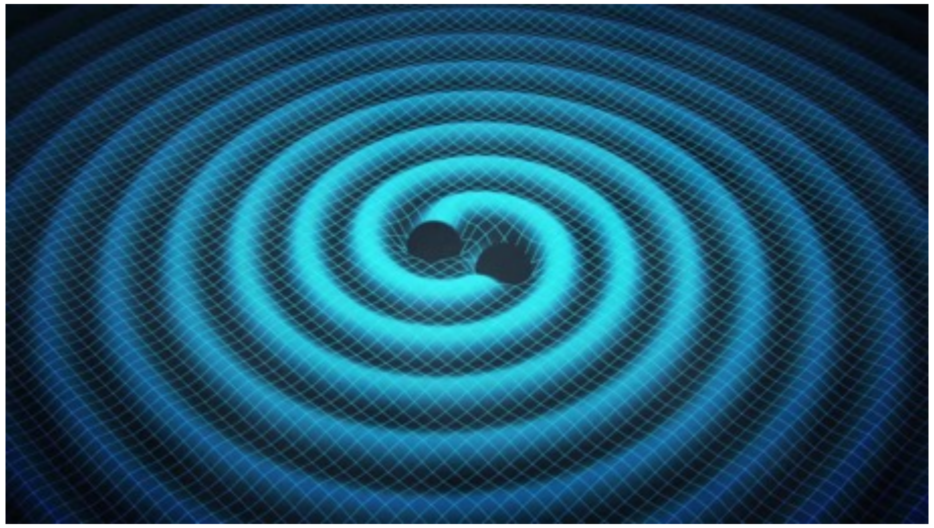
IR regularisation !

Matching equation:

$$M_L(u) = \mathrm{FP}_{B=0} \int \mathrm{d}^3x \left(\frac{|x|}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \left\{\delta_l \hat{x}_L \Sigma + \frac{\alpha_l}{c^2}\delta_{l+1}\hat{x}_{iL}\dot{\Sigma}_i + \frac{\beta_l}{c^4}\delta_{l+2}\hat{x}_{ijL}\ddot{\Sigma}_{ij}\right\}$$

The computation of M_{ij} is a PN computation in the near zone. It is known at 3PN and in progress at 4PN [2nd year PhD project]

Results



$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

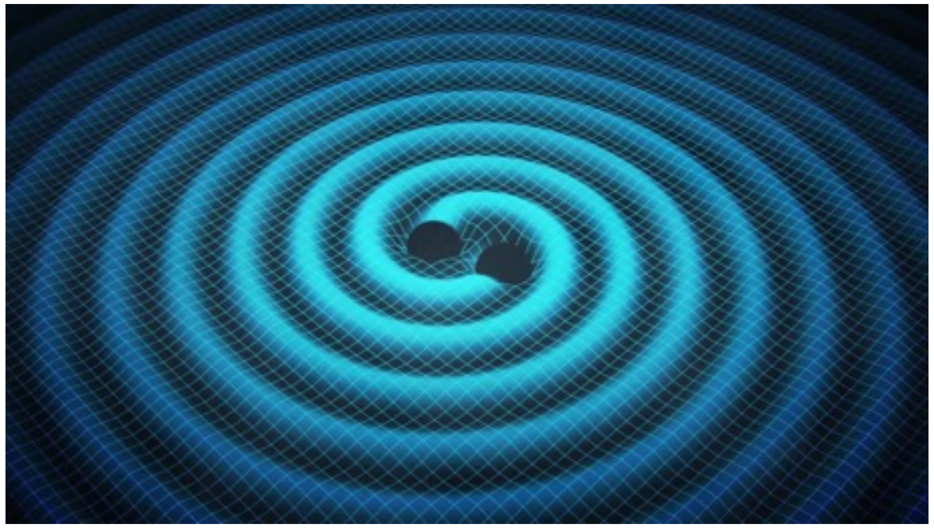
$$\gamma = \frac{Gm}{rc^2}$$

Tails

$$\mathcal{F}_{\text{quadratic}} = \frac{32c^5}{5G}\nu^2\gamma^5\Bigg\{4\pi\gamma^{3/2} + \ldots + \left(\frac{9997778801}{106444800} - \frac{6848}{105}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{8058312817}{2661120} + \frac{287}{32}\pi^2 + \frac{572}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu - \frac{12433367}{13824}\nu^2 - \frac{1026257}{266112}\nu^3\right)\pi\gamma^{9/2} + \mathcal{O}\left(\frac{1}{c^{11}}\right)\Bigg\},$$

Tails-of-tails-of-tails and Tails-of-tails × tails

$$\mathcal{F}_{\text{quartic}} = \frac{32c^5}{5G}\nu^2\gamma^5\Bigg\{\left(-\frac{467044}{3675} - \frac{3424}{105}\ln(16\gamma) + \frac{6848}{105}\ln\left(\frac{r}{r_0}\right) - \frac{6848}{105}\gamma_{\text{E}}\right)\pi\gamma^{9/2} + \mathcal{O}\left(\frac{1}{c^{11}}\right)\Bigg\}.$$



$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$x = \left(\frac{Gm\Omega}{c^3}\right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2}\right)$$

$$\begin{aligned}\mathcal{F}_{\text{total}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 \\ & + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E \right. \\ & \left. - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3 \\ & + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + (\text{unknown coefficients})x^4 \\ & + \left(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_{\text{E}} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu \right. \\ & \left. - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\right)\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}.\end{aligned}$$

Tanaka et al. gr-qc/9701050

arxiv:1607.07601

What is next?

- The 4PN equation of motion has now been computed by different methods (Hamiltonian and Fokker Lagrangian complete, EFT in progress).
- In order to compute the 4PN coefficient of the flux for circular orbits, the 4PN mass quadrupole and the 3PN current quadrupole still need to be computed.
- Once this is done, different physical results (Flux, phase evolution, waveform, polarization modes etc.) can be computed at 4PN and then directly used.

Conclusion

- With the first detections of gravitational waves and the future development of gravitational detectors, improving the accuracy of GW template is more and more crucial.
- PN computation plays a major role for building templates
- We have computed all the contributions entering the flux at 4.5PN but the 4PN coefficient is still required
- Once the 4PN mass quadrupole and the 3PN current quadrupole are computed, we'll be able to plug in all the pieces up to 4.5PN and obtain physical quantities such as the phase of the signal at 4.5PN.

Thank you